Exercise 11

Graph the particular solution and several other solutions. What characteristics do these solutions have in common?

$$y'' + 3y' + 2y = \cos x$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 3y_c' + 2y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2e^{rx}$$

Plug these formulas into equation (1).

$$r^2e^{rx} + 3(re^{rx}) + 2(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 3r + 2 = 0$$

Solve for r.

$$(r+2)(r+1) = 0$$

$$r = \{-2, -1\}$$

Two solutions to the ODE are e^{-2x} and e^{-x}. By the principle of superposition, then,

$$y_c(x) = C_1e^{-2x} + C_2e^{-x}.$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' + 3y_p' + 2y_p = \cos x \tag{2}$$

Since the inhomogeneous term is a cosine function, the particular solution is $y_p = A\cos x + B\sin x$.

$$y_p = A\cos x + B\sin x \quad \rightarrow \quad y_p' = -A\sin x + B\cos x \quad \rightarrow \quad y_p'' = -A\cos x - B\sin x$$

Substitute these formulas into equation (2).

$$(-A\cos x - B\sin x) + 3(-A\sin x + B\cos x) + 2(A\cos x + B\sin x) = \cos x$$

$$(-A + 3B + 2A)\cos x + (-B - 3A + 2B)\sin x = \cos x$$

Match the coefficients on both sides to get a system of equations for A and B.

$$\left.\begin{aligned} -A + 3B + 2A &= 1 \\ -B - 3A + 2B &= 0 \end{aligned}\right\}$$

Solving it yields

$$A = \frac{1}{10} \quad \text{and} \quad B = \frac{3}{10},$$

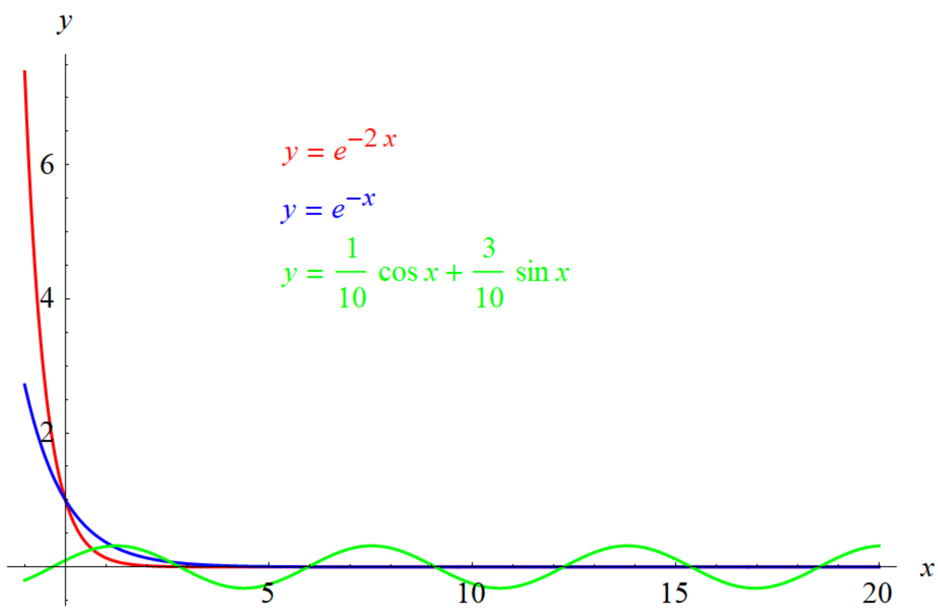
which means the particular solution is

$$y_p = \frac{1}{10}\cos x + \frac{3}{10}\sin x.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_1 e^{-2x} + C_2 e^{-x} + \frac{1}{10}\cos x + \frac{3}{10}\sin x, \end{aligned}$$

where C_1 and C_2 are arbitrary constants. Below is a graph of the two exponential functions and the particular solution.



As $x \to \infty$, all solutions tend to $(1/10)\cos x + (3/10)\sin x$. And as $x \to -\infty$, $y(x)$ blows up to infinity unless $C_1 = C_2 = 0$.